NEW OUTLOOK ON MORI THEORY, II

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Abstract. In this note we prove that all the fundamental theorems of Mori theory follow quickly from the finite generation of adjoint rings with big boundaries, which was recently proved by a self-contained argument based on extension theorems and induction on the dimension. Thus, we give a new and more efficient organization of higher dimensional algebraic geometry.

Contents

1. Introduction

The papers [Laz09, CL10] prove the following statement by a self-contained argument based on induction on the dimension and extension theorems:

Theorem 1.1 ([Laz09, Theorem A],[CL10, Theorem A])**.** *Let X be a smooth projective variety and let A be an ample $\mathbb{Q}$-divisor on X. Let Δ_i be $\mathbb{Q}$-divisors on X such that $\lfloor \Delta_i \rfloor = 0$ for $i = 1, \dots, r$, and such that the support of $\sum \Delta_i$ has simple normal crossings. Then the adjoint ring*

$$R(X; K_X + \Delta_1 + A, \dots, K_X + \Delta_r + A)$$

is finitely generated.

The definition of an adjoint ring is given in Section 2. In this short note, using Theorem 1.1, we give quick proofs of *all* the fundamental theorems of Mori theory (Rationality, Cone and Contraction theorem, existence of flips) and termination of flips with big boundary, cf. [BCHM10].

Date: 17 September 2010.

This gives a new and more efficient organization of higher dimensional algebraic geometry. In a sense, Mori theory started out with the insight that we should study curves rather than divisors. Here we demonstrate that the same theory can be built more efficiently starting from divisors after all.

It is of course tempting to state the following:

Conjecture 1.2. *Let X be a normal projective variety, and let Δ_i be $\mathbb{Q}$-divisors on X such that the pairs (X, Δ_i) are klt for $i = 1, \dots, r$. Then the adjoint ring*

$$R(X; K_X + \Delta_1, \dots, K_X + \Delta_r)$$

is finitely generated.

We hope a direct proof of this conjecture will be possible in the short term. Below we show that the following weaker conjecture implies the unconditional termination of flips with scaling and the abundance conjecture.

Conjecture 1.3. *Let (X, Δ) be a projective klt pair such that $K_X + \Delta$ is pseudo-effective, and let A be an ample $\mathbb{Q}$-divisor on X. Then the adjoint ring*

$$R(X; K_X + \Delta, K_X + \Delta + A)$$

is finitely generated.

The paper of Hu and Keel [HK00] shows awareness of some of our results here; we feel that they did not themselves work out these results explicitly because, back then, it did not seem plausible that one could prove finite generation without the Minimal Model Program.

Acknowledgements. We would like to thank P. Cascini, A.-S. Kaloghiros and M. Mustaţă for many useful conversations and comments.

2. Notation and conventions

In this paper all algebraic varieties and schemes are defined over $\mathbb{C}$. We use $\mathbb{Q}_+$ and $\mathbb{R}_+$ to denote the sets of non-negative rational and real numbers.

Let X be a normal projective variety and $\mathbf{R} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. We denote by $\mathrm{Div}_{\mathbf{R}}(X)$ the group of $\mathbf{R}$-Cartier divisors on X, and by $N^1(X)_{\mathbf{R}}$ and $N_1(X)_{\mathbf{R}}$ the groups of $\mathbf{R}$-Cartier divisors and 1-cycles on X with coefficients in $\mathbf{R}$ modulo numerical equivalence. The ample and nef cones in $N^1(X)_{\mathbb{R}}$ are denoted by $\mathrm{Amp}(X)$ and $\mathrm{Nef}(X)$. Further, $\overline{\mathrm{NE}}(X)$ denotes the closed cone of curves in $N_1(X)_{\mathbb{R}}$.

Many arguments in this paper take place inside a finite dimensional vector subspace of $\mathrm{Div}_{\mathbb{R}}(X)$; it is crucial for us to distinguish this space from $N^1(X)_{\mathbb{R}}$.

We say that an $\mathbf{R}$-divisor D is $\mathbf{R}$-*effective* if there exists a divisor $D' \geq 0$ such that $D \sim_{\mathbf{R}} D'$; that is, D has the *Iitaka dimension* $\kappa(X, D) \geq 0$. If D is $\mathbf{R}$-effective, then it is pseudo-effective. We denote by $\mathrm{Div}^{\kappa\geq 0}_{\mathbf{R}}(X)$ the set of $\mathbf{R}$-effective divisors in

$\mathrm{Div}_{\mathbf{R}}(X)$. The *stable base locus* of a $\mathbb{Q}$-effective $\mathbb{Q}$-divisor D is $\mathbf{B}(D) = \bigcap \mathrm{Bs}\,|mD|$, where the intersection is over all m sufficiently divisible.

A *pair* (X, Δ) consists of a normal projective variety X and a $\mathbb{Q}$-divisor $\Delta \geq 0$ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. When (X, Δ) is a pair, $K_X + \Delta$ is called an *adjoint divisor*.

If X is a normal projective variety with the field of fractions $k(X)$, and D an $\mathbf{R}$-divisor on X, then $\mathcal{O}_X(D) \subset k(X)$ is the sheaf given by

$$\mathcal{O}_X(D)(U) = \big\{ f \in k(X) \mid \mathrm{div}_U f + D_{|U} \geq 0 \big\}$$

for every Zariski open set $U \subset X$. We denote by $H^0(X, D)$ the group of global sections of this sheaf.

In this note we only use divisorial rings of the form

$$R = R(X; D_1, \dots, D_r) = \bigoplus_{(n_1, \dots, n_r) \in \mathbb{N}^r} H^0(X, n_1 D_1 + \dots + n_r D_r),$$

where $D_1, \dots, D_r$ are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X (not necessarily $\mathbb{Q}$-effective). The *support* of R is the cone

$$\mathrm{Supp}\, R = \left(\sum_{i=1}^{r} \mathbb{R}_+ D_i \right) \cap \mathrm{Div}_{\mathbb{R}}^{\kappa \geq 0}(X) \subset \mathrm{Div}_{\mathbb{R}}(X).$$

The choice of divisors $D_1, \dots, D_r$ gives the *tautological* linear map

$$\mathbf{D} \colon \mathbb{R}^r \ni (\lambda_1, \dots, \lambda_r) \mapsto \sum \lambda_i D_i \in \mathrm{Div}_{\mathbb{R}}(X).$$

If $\mathbb{L} \subset \mathbb{Z}^r$ is a finite index subgroup, then a ring of the form

$$R(X; \mathbf{D}_{|\mathbb{N}^r \cap \mathbb{L}}) = \bigoplus_{\mathbf{n} \in \mathbb{N}^r \cap \mathbb{L}} H^0\big(X, \mathbf{D}(\mathbf{n})\big)$$

is called a *Veronese subring of finite index* of $R(X; D_1, \dots, D_r)$. If all D_i are adjoint divisors, $R(X; D_1, \dots, D_r)$ is an *adjoint ring*.

A *geometric valuation* Γ over a normal variety X is a valuation on $k(X)$ given by the order of vanishing at the generic point of a prime divisor on some birational model $f \colon Y \to X$. If D is an $\mathbb{R}$-Cartier divisor on X, by abusing notation we use $\mathrm{mult}_\Gamma D$ to denote $\mathrm{mult}_\Gamma f^* D$.

If $\mathbf{v}, \mathbf{w}$ are vectors in a real vector space, we denote by

$$[\mathbf{v}, \mathbf{w}] = \big\{ t\mathbf{w} + (1-t)\mathbf{v} \mid t \in [0,1] \big\}$$

the closed line segment between $\mathbf{v}$ and $\mathbf{w}$. We similarly write $(\mathbf{v}, \mathbf{w})$ and $(\mathbf{v}, \mathbf{w}]$ for the open and half-open segments.

3. Simple consequences of finite generation

The following useful lemma is well-known.

Lemma 3.1. *Let X be a normal projective variety and let $D_1, \dots, D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. The ring $R = R(X; D_1, \dots, D_r)$ is finitely generated if and only if any of its Veronese subrings of finite index is finitely generated. In particular, if $D_i \sim_{\mathbb{Q}} D'_i$ and if R is finitely generated, then the ring $R' = R(X; D'_1, \dots, D'_r)$ is finitely generated.*

Proof. Let $\mathbb{L} \subset \mathbb{Z}^r$ be a subgroup of index d giving the Veronese subring $R_{\mathbb{L}}$ of R. Then for any $f \in R$ we have $f^d \in R_{\mathbb{L}}$, so R is an integral extension of $R_{\mathbb{L}}$. Furthermore, one can write an action of the group $\mathbb{Z}^r/\mathbb{L}$ on R such that $R_{\mathbb{L}}$ is the ring of invariants. Now the first claim follows from theorems of Emmy Noether on finiteness of integral closure and of ring of invariants, and the second claim follows by noting that R and R' have isomorphic Veronese subrings of finite index. $\square$

We will use the following small variation of Theorem 1.1:

Theorem 3.2. *Let X be a normal projective variety, and let Δ_i be $\mathbb{Q}$-divisors on X such that the pairs (X, Δ_i) are klt for $i = 1, \dots, r$.*

(1) If A is an ample $\mathbb{Q}$-divisor on X, then the adjoint ring

$$R(X; K_X + \Delta_1 + A, \dots, K_X + \Delta_r + A)$$

is finitely generated.

(2) If Δ_i are big, then the adjoint ring

$$R(X; K_X + \Delta_1, \dots, K_X + \Delta_r)$$

is finitely generated.

Proof. In order to prove (1), let $f\colon Y \to X$ be a log resolution of the pair $(X, \sum \Delta_i)$. For each i, let $\Gamma_i, G_i \geq 0$ be $\mathbb{Q}$-divisors on Y without common components such that G_i is f-exceptional and $K_Y + \Gamma_i = f^*(K_X + \Delta_i) + G_i$. Let $F \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor on Y such that $A' = f^*A - F$ is ample, and such that $\lfloor \Gamma'_i \rfloor = 0$ for all i, where $\Gamma'_i = \Gamma_i + F$. Then

$$R(X; K_X + \Delta_1 + A, \dots, K_X + \Delta_r + A) \simeq R(Y; K_Y + \Gamma'_1 + A', \dots, K_X + \Gamma'_r + A'),$$

and the claim follows from Theorem 1.1.

For (2), let $H \geq 0$ be an ample $\mathbb{Q}$-divisor on X such that there exist divisors $E_i \geq 0$ with $\Delta_i \sim_{\mathbb{Q}} E_i + H$. For $0 < \varepsilon \ll 1$ set $H' = \varepsilon H$ and $\Delta'_i = (1 - \varepsilon)\Delta_i + \varepsilon E_i$. Then $K_X + \Delta_i \sim_{\mathbb{Q}} K_X + \Delta'_i + H'$, and the pair $(X, \Delta'_i + H')$ is klt for every i since (X, Δ_i) is klt and $\varepsilon \ll 1$. The ring

$$R(X; K_X + \Delta'_1 + H', \dots, K_X + \Delta'_r + H')$$

is finitely generated by (1), and the claim follows from Lemma 3.1. $\square$

Remark 3.3. Let (X, Δ) be a $\mathbb{Q}$-factorial projective klt pair where Δ is big, let $\varphi\colon \operatorname{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection, and let $\|\cdot\|$ be any norm on $N^1(X)_{\mathbb{R}}$. Then there is $0 < \eta \ll 1$ with the following property: if $\|\varphi(\Delta') - \varphi(\Delta)\| < \eta$, then there exists a big divisor $\Delta'' \geq 0$ such that $K_X + \Delta' \sim_{\mathbb{Q}} K_X + \Delta''$ and (X, Δ'') is klt. Indeed, as in the proof of Theorem 3.2, there exist an ample $\mathbb{Q}$-divisor A and a $\mathbb{Q}$-divisor $B \geq 0$ such that $\Delta \sim_{\mathbb{Q}} A + B$ and the pair $(X, A + B)$ is klt. Since $\eta \ll 1$, there is an ample $\mathbb{Q}$-divisor $A' \sim A + (\Delta' - \Delta)$ such that $(X, A' + B)$ is klt, and obviously $\Delta' \sim_{\mathbb{Q}} A' + B$. Set $\Delta'' = A' + B$.

Definition 3.4. Let X be a normal projective variety, $D \in \operatorname{Div}_{\mathbb{R}}^{\kappa \geq 0}(X)$, and Γ a geometric valuation over X. The *asymptotic order of vanishing* of D along Γ is

$$o_\Gamma(D) = \inf\{\operatorname{mult}_\Gamma D' \mid D \sim_{\mathbb{R}} D' \geq 0\}.$$

If X is a normal projective variety, D an integral divisor on X with $|D| \neq \emptyset$, and Γ a geometric valuation over X, we write

$$\operatorname{mult}_\Gamma |D| = \min_{D' \in |D|} \operatorname{mult}_\Gamma D'.$$

We use the following result without explicit mention.

Lemma 3.5. *Let X be a normal projective variety, let $D \in \operatorname{Div}_{\mathbb{Q}}^{\kappa \geq 0}(X)$, and let Γ be a geometric valuation over X. Then $o_\Gamma(D) = \inf \frac{1}{p} \operatorname{mult}_\Gamma |pD|$ for all sufficiently divisible positive integers p.*

Proof. There exist an $\mathbb{R}$-divisor $F \geq 0$, real numbers $r_1, \dots, r_k$ and rational functions $f_1, \dots, f_k \in k(X)$ such that $F = D + \sum_{i=1}^k r_i(f_i)$. Let $W \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ be the subspace spanned by the components of D, D' and all (f_i), let $\|\cdot\|$ be the sup-norm on W, and let $W_0 \subseteq W$ be the subspace of divisors $\mathbb{R}$-linearly equivalent to zero. Note that W_0 is a rational subspace of W, and consider the quotient map $\pi\colon W \to W/W_0$. Then the set $\mathcal{G} = \{G \in \pi^{-1}(\pi(D)) \mid G \geq 0\}$ is not empty as it contains F, and it is cut out from W by rational hyperplanes. Thus, for every $\varepsilon > 0$, $\mathcal{G}$ contains a $\mathbb{Q}$-divisor $D' \geq 0$ such that $D \sim_{\mathbb{Q}} D'$ and $\|D' - D\| < \varepsilon$. This proves the lemma. □

Theorem 3.6. *Let X be a normal projective variety and let $D_1, \dots, D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Assume that the ring $R = R(X; D_1, \dots, D_r)$ is finitely generated, and let $\mathbf{D}\colon \mathbb{R}^r \ni (\lambda_1, \dots, \lambda_r) \mapsto \sum \lambda_i D_i \in \operatorname{Div}_{\mathbb{R}}(X)$ be the tautological map.*

(1) The support of R is a rational polyhedral cone.

(2) Suppose that $\operatorname{Supp} R \ni B$, where B is a big divisor. If $D \in \sum \mathbb{R}_+ D_i$ is pseudo-effective, then $D \in \operatorname{Supp} R$.

(3) There is a finite rational polyhedral subdivision $\operatorname{Supp} R = \bigcup \mathcal{C}_i$ such that, for every geometric valuation Γ over X, o_Γ is linear on $\mathcal{C}_i$. Furthermore, there is the coarsest *subdivision with this property, in the sense that, if i and j are distinct, there is at least one geometric valuation Γ over X such that (the linear extensions of) $o_\Gamma|_{\mathcal{C}_i}$ and $o_\Gamma|_{\mathcal{C}_j}$ are different.*

(4) *There is a finite index subgroup* $\mathbb{L} \subset \mathbb{Z}^r$ *such that for all* $\mathbf{n} \in \mathbb{N}^r \cap \mathbb{L}$, *if* $\mathbf{D}(\mathbf{n}) \in \operatorname{Supp} R$, *then*

$$o_\Gamma\big(\mathbf{D}(\mathbf{n})\big) = \operatorname{mult}_\Gamma |\mathbf{D}(\mathbf{n})|$$

for all geometric valuations Γ *over* X.

Proof. Claim (1) is obvious. For (2), observe that every divisor in $(D, B]$ is big, hence $(D, B] \subset \operatorname{Supp} R$. But then $[D, B] \subset \operatorname{Supp} R$ since $\operatorname{Supp} R$ is closed by (1).

Claims (3) and (4) can be extracted verbatim from the proof of [ELM+06, Theorem 4.1]. Note that the statement of that result carries the assumption that $\operatorname{Supp} R$ contains a big divisor, but it is not necessary for the proof here. Consider the system of ideals $(\mathfrak{b}_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^r}$, where $\mathfrak{b}_{\mathbf{n}}$ is the base ideal of the linear system $|\mathbf{D}(\mathbf{n})|$. This is a finitely generated system, so by [ELM+06, Proposition 4.7] there is a rational polyhedral subdivision $\mathbb{R}^r_+ = \bigcup \mathcal{D}_i$ and a positive integer d such that for every i, if $e^i_1, \dots, e^i_s$ are generators of $\mathbb{N}^r \cap \mathcal{D}_i$, then

$$\overline{\mathfrak{b}_{d\sum_j p_j e^i_j}} = \overline{\textstyle\prod_j \mathfrak{b}^{p_j}_{de^i_j}}$$

for every $(p_1, \dots, p_s) \in \mathbb{N}^s$. Since a valuation of an ideal is equal to that of its integral closure, we deduce that for every geometric valuation Γ over X, o_Γ is linear on each of the cones $\mathcal{C}_i = \operatorname{Supp} R \cap \mathbf{D}(\mathcal{D}_i)$, and we can take $\mathbb{L} = (d\mathbb{Z})^r$. The existence of the coarsest subdivision as in (3) follows directly from convexity of asymptotic order functions. □

The following statement forms part of [Laz09, Theorem C] and [CL10, Theorem B]; here we prove it as an easy consequence of Theorem 3.2.

Corollary 3.7. *Let* (X, Δ) *be a projective klt pair where* Δ *is big. If* $K_X + \Delta$ *is pseudo-effective, then it is* $\mathbb{Q}$*-effective.*

Proof. Let A be an ample $\mathbb{Q}$-divisor on X such that the pair $(X, \Delta + A)$ is klt and $K_X + \Delta + A$ is also ample. By Theorem 3.2, the adjoint ring

$$R = R(X; K_X + \Delta, K_X + \Delta + A)$$

is finitely generated and, by construction, $\operatorname{Supp} R$ contains the big divisor $K_X + \Delta + A$. The conclusion now follows from Theorem 3.6(2). □

Lemma 3.8. *Let* X *be a normal projective variety and* $D \in \operatorname{Div}^{\kappa\geq 0}_{\mathbb{Q}}(X)$.

(1) *If* D *is semiample, then* $o_\Gamma(D) = 0$ *for every geometric valuation* Γ *over* X.
(2) *Assume that* $R(X, D)$ *is finitely generated. If* $o_\Gamma(D) = 0$ *for every geometric valuation* Γ *over* X, *then* D *is semiample.*

Proof. Assume D is semiample. Then a positive integer multiple pD is basepoint free, thus clearly all $o_\Gamma(D) = 0$.

Conversely, if $R(X, D)$ is finitely generated and $o_\Gamma(D) = 0$ for a valuation Γ, then the centre of Γ on X is not in $\mathbf{B}(D)$ by Theorem 3.6(4). Since every point on X is the centre of some valuation Γ, we have $\mathbf{B}(D) = \emptyset$ and thus D is semiample. □

Next we derive a special case of Kawamata's Basepoint free theorem as a consequence of Theorem 1.1.

Corollary 3.9. *Let (X,Δ) be a projective klt pair where Δ is big. If $K_X+\Delta$ is nef, then it is semiample.*

Proof. Let A be an ample $\mathbb{Q}$-divisor on X such that the pair $(X,\Delta+A)$ is klt. By Theorem 3.2, the ring

$$R=R(X;K_X+\Delta,K_X+\Delta+A)$$

is finitely generated, and $\operatorname{Supp}R=\mathbb{R}_+(K_X+\Delta)+\mathbb{R}_+(K_X+\Delta+A)$ by Theorem 3.6(2). For each $\varepsilon>0$, the divisor $K_X+\Delta+\varepsilon A$ is ample, thus $o_\Gamma(K_X+\Delta+\varepsilon A)=0$ for every geometric valuation Γ over X. Therefore, all o_Γ are identically zero on $\operatorname{Supp}R$ by Theorem 3.6(3), and thus $K_X+\Delta$ is semiample by Lemma 3.8(2). $\square$

4. Rationality, Cone and Contraction theorem

Definition 4.1. Let W be a finite dimensional real vector space, $\mathcal{C}\subset W$ a closed cone, and $\mathbf{v}\in W$. The *visible boundary* of $\mathcal{C}$ from $\mathbf{v}$ is the set

$$V=\big\{\mathbf{w}\in\partial\mathcal{C}\mid[\mathbf{v},\mathbf{w}]\cap\mathcal{C}=\{\mathbf{w}\}\big\}.$$

The following statement is a reformulation of the Rationality, Cone and Contraction theorem due to Kawamata [Kaw09]. Here, relint denotes the relative interior.

Theorem 4.2 (Rationality, Cone and Contraction theorem)**.** *Let (X,Δ) be a projective klt pair. Let V be the visible boundary of* $\operatorname{Nef}(X)$ *from the class $\mathbf{v}_0\in N^1(X)_\mathbb{R}$ of the divisor $K_X+\Delta$. Then:*

1. *every compact subset $F\subset\operatorname{relint}V$ is contained in a union of finitely many supporting rational hyperplanes;*
2. *every $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X with class in* $\operatorname{relint}V$ *is semiample.*

Remark 4.3. Let L be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with class in $\operatorname{relint}V$. By the theorem, L is semiample; in other words, there is a contraction morphism

$$f_L\colon X\to Y$$

and $L=f_L^*A$ for an ample $\mathbb{Q}$-divisor A on Y.

If $F_L\subset V$ is the smallest face containing L, then we have natural identifications

$$\operatorname{Nef}(Y)=F_L\subset N^1(Y)_\mathbb{R}=\langle F_L\rangle\subset N^1(X)_\mathbb{R},$$

where $\langle F_L\rangle$ denotes the vector subspace generated by F_L.

If L and M are $\mathbb{Q}$-divisors in $\operatorname{relint}V$, and $F_L\subset F_M$, then there is a factorization $f_L=g\circ f_M$. In particular, $M=f_M^*(D)$, where $D=g^*(A)$ is a semiample divisor.

The f_L are contractions of faces of $\overline{\mathrm{NE}}(X)$ in Mori theory; extremal contractions correspond to those L that lie in the relative interiors of faces of maximal dimension.

Proof of Theorem 4.2. We work with $N^1(X)_\mathbb{R}$ equipped with the sup norm.

We first show (1). If $\mathbf{v}_0 \in \operatorname{Nef}(X)$, there is nothing to prove; thus, we can and will assume that $\mathbf{v}_0 \notin \operatorname{Nef}(X)$. Consider the cone $\mathcal{C} = \mathbb{R}_+\mathbf{v}_0 + \operatorname{Nef}(X)$; by compactness of F, there is a rational number $0 < \varepsilon \ll 1$ and finitely many rational points $\mathbf{u}_1, \dots, \mathbf{u}_p \in \operatorname{int} \mathcal{C}$ such that $F \subset \operatorname{int}\big(\bigcup B(\mathbf{u}_i, \varepsilon)\big) \subset \operatorname{int} \mathcal{C}$, where $B(\mathbf{u}_i, \varepsilon)$ denotes the closed ball. Since we are working in the sup norm, $B(\mathbf{u}_i, \varepsilon)$ are cubes, thus the convex hull $\mathcal{B}$ of $\bigcup B(\mathbf{u}_i, \varepsilon)$ is a rational polytope: denote by $\mathbf{w}_1, \dots, \mathbf{w}_m$ its vertices. Then

$$\mathbf{w}_j \in \operatorname{int} \mathcal{C} = \bigcup_{\mathbf{a} \in \operatorname{Amp}(X)} (\mathbf{v}_0, \mathbf{a}),$$

so there exist rational ample classes $\mathbf{a}_j$ and rational numbers $t_j \in (0,1)$ such that $\mathbf{w}_j = t_j\mathbf{v}_0 + (1-t_j)\mathbf{a}_j$. For each j, choose an ample $\mathbb{Q}$-divisor A_j with class $\frac{1-t_j}{t_j}\mathbf{a}_j$ such that the pair $(X, \Delta + A_j)$ is klt; then $\mathbf{w}_j$ is the class of the divisor $t_j(K_X + \Delta + A_j)$. By Theorem 3.2, the adjoint ring

$$R = R(X; K_X + \Delta + A_1, \dots, K_X + \Delta + A_m)$$

is finitely generated, and denote by $\varphi\colon \sum \mathbb{R}_+(K_X + \Delta + A_j) \to N^1(X)_\mathbb{R}$ the natural projection; by construction, $F \subset \varphi\big(\sum \mathbb{R}_+(K_X + \Delta + A_j)\big)$. Then $\mathcal{L} = \operatorname{Supp} R$ is a rational polyhedral cone by Theorem 3.6(1), and since $F \subset \operatorname{int} \mathcal{B} \cap \partial \operatorname{Nef}(X)$, we have $\mathcal{B} \cap \operatorname{Amp}(X) \neq \emptyset$, so $\mathcal{L}$ contains ample divisors. Therefore $\varphi^{-1}(F) \subset \mathcal{L}$ by Theorem 3.6(2), as every divisor with class in F is pseudo-effective.

Let $\mathcal{L} = \bigcup \mathcal{L}_k$ be the coarsest subdivision as in Theorem 3.6(3). Then there exists k such that $\mathcal{L}_k \cap \varphi^{-1}\big(\operatorname{Amp}(X)\big) \neq \emptyset$, and we claim that actually $\mathcal{L}_k = \mathcal{L} \cap \varphi^{-1}\big(\operatorname{Nef}(X)\big)$. This immediately implies (1): since $\varphi^{-1}(F) \subset \mathcal{L} \cap \varphi^{-1}\big(\operatorname{Nef}(X)\big)$, then by the claim $\varphi^{-1}(F) \subset \partial \mathcal{L}_k$, hence the result.

To show the claim, note that by Theorem 3.6(3) all asymptotic order functions o_Γ are identically zero on $\mathcal{L}_k$, because they are so on the nonempty subset $\mathcal{L}_k \cap \varphi^{-1}\big(\operatorname{Amp}(X)\big)$; therefore, by Lemma 3.8(2), every element of $\mathcal{L}_k$ is semiample and thus $\mathcal{L}_k \subset \varphi^{-1}\big(\operatorname{Nef}(X)\big)$. Conversely, it is clear that all asymptotic order functions are identically zero on $\mathcal{L} \cap \varphi^{-1}\big(\operatorname{Amp}(X)\big)$. Therefore, by the coarseness of the subdivision, $\mathcal{L} \cap \varphi^{-1}\big(\operatorname{Amp}(X)\big)$ is entirely contained in $\mathcal{L}_k$, and then also $\mathcal{L} \cap \varphi^{-1}\big(\operatorname{Nef}(X)\big) \subset \mathcal{L}_k$ since $\mathcal{L}_k$ is closed.

We now show (2). Let D be a $\mathbb{Q}$-divisor on X with class $\mathbf{v} \in \operatorname{relint} V$. Similarly as above, there is a rational ample class $\mathbf{a}$ and rational number $t \in (0,1)$ such that $\mathbf{v} = t\mathbf{v}_0 + (1-t)\mathbf{a}$. Then we can choose an ample $\mathbb{Q}$-divisor A with class $\frac{1-t}{t}\mathbf{a}$ such that the pair $(X, \Delta + A)$ is klt and

$$D \sim_\mathbb{Q} t(K_X + \Delta + A).$$

Now D is semiample by Corollary 3.9. $\square$

5. Birational contractions

A birational map $f\colon X \dashrightarrow Y$ between normal varieties is a *birational contraction* if f^{-1} does not contract divisors. If additionally X and Y are $\mathbb{Q}$-factorial, and $(p,q)\colon W \to X \times Y$ is a resolution of f, then we define the map $f^*\colon \mathrm{Div}_{\mathbb{R}}(Y) \to \mathrm{Div}_{\mathbb{R}}(X)$ as $f^* = p_* \circ q^*$; this does not depend on the choice of W. Note that $f^* = f_*^{-1}$ when f is an isomorphism in codimension 1. Extremal contractions and flips are examples of birational contractions.

Lemma 5.1. *Let X and Y be $\mathbb{Q}$-factorial projective varieties, let $f\colon X \dashrightarrow Y$ be a birational contraction, and let $\tilde{f}\colon k(X) \simeq k(Y)$ be the induced isomorphism. Then:*

(1) $f_* \operatorname{div}_X \varphi = \operatorname{div}_Y \tilde{f}(\varphi)$ for every $\varphi \in k(X)$;
(2) for every geometric valuation Γ on $k(X)$ and for every $\varphi \in k(X)$ we have $\operatorname{mult}_\Gamma(\operatorname{div}_X \varphi) = \operatorname{mult}_\Gamma\big(\operatorname{div}_Y \tilde{f}(\varphi)\big)$;
(3) if f is an isomorphism in codimension one, then $f_*\colon \mathrm{Div}_{\mathbb{R}}(X) \to \mathrm{Div}_{\mathbb{R}}(Y)$ is an isomorphism, and for every $D \in \mathrm{Div}_{\mathbb{R}}(X)$ the map $\tilde{f}$ restricts to the isomorphism $H^0(X,D) \simeq H^0(Y,f_*D)$.

Proof. For (1), let $U \subset X$ and $V \subset Y$ be open subsets such that $f_{|U}\colon U \to V$ is an isomorphism and $\operatorname{codim}_Y(Y \setminus V) \geq 2$. Then obviously $(f_* \operatorname{div}_X \varphi)_{|V} = (\operatorname{div}_Y \tilde{f}(\varphi))_{|V}$, thus the claim. The second claim is easily verified on a common resolution of X and Y. If additionally $\operatorname{codim}_X(X \setminus U) \geq 2$, then (3) follows from $H^0(U,D) \simeq H^0(V,f_*D)$ since X and Y are normal. □

The following lemma will be used in the proof of termination with scaling.

Lemma 5.2. *Let X and Y be $\mathbb{Q}$-factorial projective varieties and let $f\colon X \dashrightarrow Y$ be a birational map which is an isomorphism in codimension one. Let $\mathcal{C} \subset \mathrm{Div}_{\mathbb{R}}^{\kappa\geq 0}(X)$ be a cone, and fix a geometric valuation Γ over X. Then the asymptotic order of vanishing o_Γ is linear on $\mathcal{C}$ if and only if it is linear on $f_*\mathcal{C} \subset \mathrm{Div}_{\mathbb{R}}(Y)$.*

Proof. For every $D \in \mathcal{C}$, denote $V_D = \{D_X - D \mid D \sim_{\mathbb{R}} D_X \text{ and } D_X \geq 0\} \subset \mathrm{Div}_{\mathbb{R}}(X)$ and $W_D = \{D_Y - f_*D \mid f_*D \sim_{\mathbb{R}} D_Y \text{ and } D_Y \geq 0\} \subset \mathrm{Div}_{\mathbb{R}}(Y)$. Note that the elements of V_D and W_D are $\mathbb{R}$-linear combinations of principal divisors. By Lemma 5.1 we have an isomorphism $f_*|_{V_D}\colon V_D \simeq W_D$, and $\operatorname{mult}_\Gamma P_X = \operatorname{mult}_\Gamma f_*P_X$ for every $P_X \in V_D$. Therefore

$$o_\Gamma(D) - \operatorname{mult}_\Gamma D = \inf_{P_X \in V_D} \operatorname{mult}_\Gamma P_X = \inf_{P_X \in V_D} \operatorname{mult}_\Gamma f_*P_X = o_\Gamma(f_*D) - \operatorname{mult}_\Gamma f_*D,$$

hence the function $o_\Gamma(\cdot) - o_\Gamma\big(f_*(\cdot)\big)\colon \mathcal{C} \to \mathbb{R}$ is equal to the linear map $\operatorname{mult}_\Gamma(\cdot) - \operatorname{mult}_\Gamma f_*(\cdot)$. The claim now follows. □

The following lemma is well-known; for this particular formulation and a proof, see [HK00, Lemma 1.7].

Lemma 5.3. *Let $f\colon X \dashrightarrow Y$ and $g\colon X \dashrightarrow Z$ be birational contractions. Suppose that there exist an ample divisor A on Y and a nef divisor B on Z such that*

$$f^*A + F = g^*B + G,$$

where $F \geq 0$ is f-exceptional and $G \geq 0$ is g-exceptional. Then the birational map $f \circ g^{-1}\colon Z \dashrightarrow Y$ is a morphism. □

The following is an easy consequence of the property of separatedness of schemes; see for instance [Har77, Ex. II.4.2].

Lemma 5.4. *Let X be a reduced scheme, let Y be a separated scheme, and let f and g be two morphisms from X to Y. Assume that $f_{|U} = g_{|U}$ on a Zariski dense open subset $U \subset X$. Then $f = g$.* □

6. Termination with scaling

Definition 6.1. Let (X, Δ) be a projective klt pair and let A be a big $\mathbb{Q}$-divisor on X such that $K_X + \Delta + A$ is nef. The *nef threshold* of (X, Δ) with respect to A is

$$\lambda = \lambda(X, \Delta, A) = \inf\{t \in \mathbb{R}_+ \mid K_X + \Delta + tA \text{ is nef}\}.$$

The following lemma is a central ingredient in the Minimal Model Program with scaling, to be discussed shortly.

Lemma 6.2. *Let (X, Δ) be a $\mathbb{Q}$-factorial projective klt pair such that $K_X + \Delta$ is not nef, and let A be a big $\mathbb{Q}$-divisor on X such that $(X, \Delta + A)$ is klt and $K_X + \Delta + A$ is nef. Let $\lambda = \lambda(X, \Delta, A)$ be the nef threshold. Then $\lambda \in \mathbb{Q}_+$, and there exists an extremal ray $R \subset \overline{\mathrm{NE}}(X)$ with $(K_X + \Delta + \lambda A) \cdot R = 0$ and $(K_X + \Delta) \cdot R < 0$.*

Proof. Denote by $\varphi\colon \operatorname{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection and let $\|\cdot\|$ be any norm on $N^1(X)_{\mathbb{R}}$. Pick finitely many big $\mathbb{Q}$-divisors $\Delta_1, \ldots, \Delta_r$ such that:

(1) $\|\varphi(\Delta + \lambda A) - \varphi(\Delta_i)\| \ll 1$ for all i;
(2) writing $\mathcal{C} = \sum_{i=1}^r \mathbb{R}_+(K_X + \Delta_i) \subset \operatorname{Div}_{\mathbb{R}}(X)$, we have $K_X + \Delta + \lambda A \in \operatorname{int} \mathcal{C}$, and the dimension of the cone $\varphi(\mathcal{C}) \subset N^1(X)_{\mathbb{R}}$ is $\dim N^1(X)_{\mathbb{R}}$.

By Theorem 3.2 and Remark 3.3, the ring

$$R = R(X; K_X + \Delta_1, \ldots, K_X + \Delta_r)$$

is finitely generated. An argument similar to that in the proof of Theorem 4.2 shows that the cone $\operatorname{Supp} R \cap \varphi^{-1}\big(\operatorname{Nef}(X)\big)$ is rational polyhedral, and that there is a rational codimension one face $F \ni \varphi(K_X + \Delta + \lambda A)$ of $\operatorname{Nef}(X)$. This implies $\lambda \in \mathbb{Q}_+$, and we choose $R \subset \overline{\mathrm{NE}}(X)$ to be the extremal ray dual to F. □

The Minimal Model Program with scaling. Let (X,Δ) be a $\mathbb{Q}$-factorial projective klt pair, and A a big $\mathbb{Q}$-divisor on X. Assume that $(X,\Delta+A)$ is klt and $K_X+\Delta+A$ is nef. The Minimal Model Program with scaling of A is the following version of the Minimal Model Program for $K_X+\Delta$. Starting with $X_1=X$, $\Delta_1=\Delta$ and $A_1=A$, we define an inductive sequence of rational maps

$$(X_1,\Delta_1+\lambda_1A_1)\overset{f_1}{\dashrightarrow}\cdots\overset{f_{i-1}}{\dashrightarrow}(X_i,\Delta_i+\lambda_iA_i)\overset{f_i}{\dashrightarrow}(X_{i+1},\Delta_{i+1}+\lambda_{i+1}A_{i+1})\overset{f_{i+1}}{\dashrightarrow}\cdots,$$

where Δ_i, A_i are the proper transforms of Δ, A on X_i, $\lambda_i=\lambda(X_i,\Delta_i,A_i)$ is the nef threshold, and $f_i\colon X_i\dashrightarrow X_{i+1}$ is the extremal contraction or the flip corresponding to a $(K_{X_i}+\Delta_i)$-extremal ray R_i with $(K_{X_i}+\Delta_i+\lambda_iA_i)\cdot R_i=0$ as in Lemma 6.2. Note that $K_{X_i}+\Delta_i+\lambda_{i-1}A_i$ is nef by Remark 4.3, thus the sequence λ_i is non-increasing.

Remark 6.3 (Existence of flips)**.** If $R\subset\overline{\mathrm{NE}}(X)$ is an extremal ray such that $(K_X+\Delta)\cdot R<0$, then the existence of the contraction morphism $f_R\colon X\to Y$ follows from Theorem 4.2. If the contraction is small, then the existence of the flip is equivalent to the finite generation of the relative adjoint ring

$$R(X/Y,K_X+\Delta)=\bigoplus_{n\geq0}(f_R)_*\mathcal{O}_X\big(n(K_X+\Delta)\big).$$

We may assume that Y is affine, and then the finite generation follows from that of the canonical ring $R(X,K_X+\Delta)$; this follows from Theorem 1.1, see [CL10, Theorem 1.1]. Now it is a well-known fact that $\operatorname{Proj}_Y R(X/Y,K_X+\Delta)$ is the flip.

Remark 6.4. Let (X,Δ) be a projective klt pair, and let $f\colon X\dashrightarrow Y$ be a composition of $(K_X+\Delta)$-divisorial contractions and $(K_X+\Delta)$-flips. Then by [KM98, Lemma 3.38], for every resolution $(p,q)\colon W\to X\times Y$ of f we have

$$p^*(K_X+\Delta)=q^*(K_Y+f_*\Delta)+E,$$

where $E\geq0$ is a q-exceptional divisor such that $E\neq0$. Therefore the map f cannot be an isomorphism, and the formula above implies

$$H^0(X,K_X+\Delta)\simeq H^0(Y,K_Y+f_*\Delta).$$

Now we can establish termination of flips with scaling with big boundary.

Theorem 6.5. *Let (X_1,Δ_1) be a projective $\mathbb{Q}$-factorial klt pair where Δ_1 is big. Let A_1 be a big $\mathbb{Q}$-divisor on X such that (X_1,Δ_1+A_1) is klt and $K_{X_1}+\Delta_1+A_1$ is nef, and let $\lambda_1=\lambda(X_1,\Delta_1,A_1)$. Then there is no infinite sequence*

$$(X_1,\Delta_1+\lambda_1A_1)\overset{f_1}{\dashrightarrow}\cdots\overset{f_{i-1}}{\dashrightarrow}(X_i,\Delta_i+\lambda_iA_i)\overset{f_i}{\dashrightarrow}(X_{i+1},\Delta_{i+1}+\lambda_{i+1}A_{i+1})\overset{f_{i+1}}{\dashrightarrow}\cdots$$

of flips of the Minimal Model Program with scaling of A_1.

Proof. Assume that such an infinite sequence exists; then in particular $\lambda_i>0$ for all i. Denote by $\varphi\colon\operatorname{Div}_{\mathbb{R}}(X_1)\to N^1(X_1)_{\mathbb{R}}$ the natural projection and let $\|\cdot\|$ be any norm on $N^1(X_1)_{\mathbb{R}}$. Choose big $\mathbb{Q}$-divisors $H_1,\dots,H_r$ on X_1 such that:

(1) $\|\varphi(\Delta_1 + \lambda_1 A_1) - \varphi(H_j)\| \ll 1$ for all j;
(2) writing $\mathcal{C}^1 = \mathbb{R}_+(K_{X_1} + \Delta_1) + \sum_{j=1}^r \mathbb{R}_+(K_{X_1} + H_j) \subset \operatorname{Div}_{\mathbb{R}}(X_1)$, we have $K_{X_1} + \Delta_1 + \lambda_1 A_1 \in \operatorname{int} \mathcal{C}^1$, and the dimension of the cone $\varphi(\mathcal{C}^1) \subset N^1(X_1)_{\mathbb{R}}$ is $\dim N^1(X_1)_{\mathbb{R}}$.

For each i, let H_j^i be the proper transforms of H_j on X_i, and write

$$R_i = R(X_i; K_{X_i} + \Delta_i, K_{X_i} + H_1^i, \dots, K_{X_i} + H_r^i).$$

By Lemma 5.1(3) we have $R_i \simeq R_1$ for all i, and these rings are finitely generated by Theorem 3.2 and Remark 3.3.

By construction, the cone $\mathcal{C}^1$ contains an open neighbourhood of the nef divisor $K_{X_1} + \Delta_1 + \lambda_1 A_1$, so it contains ample divisors in its interior, and thus the cone $\varphi(\operatorname{Supp} R_1) \subset N^1(X_1)_{\mathbb{R}}$ also has dimension $\dim N^1(X_1)_{\mathbb{R}}$. Let $\operatorname{Supp} R_1 = \bigcup \mathcal{C}_k^1$ be the finite rational polyhedral subdivision as in Theorem 3.6(3).

Let us denote by $\mathcal{C}_k^i \subset \operatorname{Div}_{\mathbb{R}}(X_i)$ the proper transform of $\mathcal{C}_k^1$ and by $\mathcal{C}^i \subset \operatorname{Div}_{\mathbb{R}}(X_i)$ the proper transform of $\mathcal{C}^1$. By Lemma 5.2, for every geometric valuation Γ the asymptotic order function o_Γ is linear on each $\mathcal{C}_k^i$.

By construction, if $0 < \lambda \le \lambda_1$, then $K_{X_1} + \Delta_1 + \lambda A_1 \in \operatorname{int} \mathcal{C}^1$, so $K_{X_i} + \Delta_i + \lambda_i A_i \in \operatorname{int} \mathcal{C}^i$ for every i. Since $K_{X_i} + \Delta_i + \lambda_i A_i$ is nef, we have $K_{X_i} + \Delta_i + \lambda_i A_i \in \operatorname{Supp} R_i$ by Corollary 3.7. Hence, as in the proof of Theorem 4.2, for each i there exists an index k such that the image of $\mathcal{C}_k^i$ in $N^1(X_i)_{\mathbb{R}}$ is a subset of $\operatorname{Nef}(X_i)$. Therefore

$$\varphi(\mathcal{C}_k^1) \subset (f_{i-1} \circ \cdots \circ f_1)^* \operatorname{Nef}(X_i).$$

Since there are finitely many cones $\mathcal{C}_k^1$, there are two indices p and q such that the cones $(f_{p-1} \circ \cdots \circ f_1)^* \operatorname{Nef}(X_p)$ and $(f_{q-1} \circ \cdots \circ f_1)^* \operatorname{Nef}(X_q)$ share a common interior point. Thus, by Lemma 5.3 the map $X_p \dashrightarrow X_q$ is biregular. But then, by Lemma 5.4, it is an isomorphism, which is a contradiction by Remark 6.4. $\square$

Corollary 6.6. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair where Δ is big, and let A be an ample $\mathbb{Q}$-divisor on X. Then:*

(1) if $K_X + \Delta$ is pseudo-effective, the Minimal Model Program with scaling of A terminates with a minimal model, and the canonical model of (X, Δ) exists;
(2) if $K_X + \Delta$ is not pseudo-effective, the Minimal Model Program with scaling of A terminates with a Mori fibre space. $\square$

Corollary 6.7. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair such that $K_X + \Delta$ is not pseudo-effective, and let A be an ample $\mathbb{Q}$-divisor on X. Then the Minimal Model Program with scaling of A terminates with a Mori fibre space.*

Proof. There exists $0 < \mu \ll 1$ such that $K_X + \Delta + \mu A$ is also not pseudo-effective, thus all $(K_X + \Delta)$-extremal contractions are $(K_X + \Delta + \mu A)$-extremal contractions. We conclude by Corollary 6.6. $\square$

Finally we show that Conjecture 1.3 implies unconditional Minimal Model Program with scaling.

Theorem 6.8. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair, and let A be an ample $\mathbb{Q}$-divisor on X. If Conjecture 1.3 holds, then the Minimal Model Program with scaling of A terminates.*

Proof. By Corollary 6.7 we can assume that $K_X+\Delta$ is pseudo-effective. Consider the Minimal Model Program with scaling of $A_1 = A$ starting from the pair $(X_1, \Delta_1) = (X, \Delta)$:

$$(X_1, \Delta_1 + \lambda_1 A_1) \overset{f_1}{\dashrightarrow} \cdots \overset{f_{i-1}}{\dashrightarrow} (X_i, \Delta_i + \lambda_i A_i) \overset{f_i}{\dashrightarrow} (X_{i+1}, \Delta_{i+1} + \lambda_{i+1} A_{i+1}) \overset{f_{i+1}}{\dashrightarrow} \cdots,$$

and consider the adjoint rings

$$R_i = R(X_i; K_{X_i} + \Delta_i, K_{X_i} + \Delta_i + \lambda_i A_i).$$

By Remark 6.4 we have $R_i \simeq R(X_1; K_{X_1} + \Delta_1, K_{X_1} + \Delta_1 + \lambda_i A_1)$, thus all R_i are finitely generated by Conjecture 1.3. Note that here we invoke Conjecture 1.3 on X_1 and not on X_i, since the divisor A_i is, in general, only big and not ample.

Assume that there exists an index i_0 and a sequence of flips f_i for $i \geq i_0$. We will show that this sequence is finite, and thus the program terminates.

By Theorem 3.6(2) we have

$$\operatorname{Supp} R_{i_0} = \mathbb{R}_+(K_{X_{i_0}} + \Delta_{i_0}) + \mathbb{R}_+(K_{X_{i_0}} + \Delta_{i_0} + \lambda_{i_0} A_{i_0}).$$

Let $\operatorname{Supp} R_{i_0} = \bigcup \mathcal{C}_k^{i_0}$ be a rational polyhedral subdivision as in Theorem 3.6(3), and let $\mathcal{C}_k^i \subset \operatorname{Div}_{\mathbb{R}}(X_i)$ denote the proper transform of $\mathcal{C}_k^{i_0}$ for $i \geq i_0$. By Lemma 5.2, for each geometric valuation Γ on $k(X_i)$, the function o_Γ is linear on $\mathcal{C}_k^i$.

Assume that, for some index k, $K_{X_i} + \Delta_i + \lambda_i A_i \in \operatorname{int} \mathcal{C}_k^i$. By Corollary 3.9, $K_{X_i} + \Delta_i + \lambda_i A_i$ is semiample, hence $o_\Gamma(K_{X_i} + \Delta_i + \lambda_i A_i) = 0$ for every geometric valuation Γ over X_i. Since the functions o_Γ are linear and non-negative on $\mathcal{C}_k^i$, they all must then be identically zero on $\mathcal{C}_k^i$. Then, by Lemma 3.8(2) every divisor in $\mathcal{C}_k^i$ is semiample, and thus nef, so it can not be true that λ_i is *smallest* such that $K_{X_i} + \Delta_i + \lambda_i A_i$ is nef, a contradiction.

Thus, if $K_{X_i} + \Delta_i + \lambda_i A_i$ is in $\mathcal{C}_k^i$, it is on one of the two boundary rays of $\mathcal{C}_k^i$, and therefore $K_{X_{i_0}} + \Delta_{i_0} + \lambda_i A_{i_0}$ is on one of the two boundary rays of $\mathcal{C}_k^{i_0}$. Since there are finitely many cones $\mathcal{C}_k^{i_0}$, the set $\{\lambda_i\}$ is finite. Write $\lambda = \min\{\lambda_i\}$. If $\lambda = 0$, then $K_{X_i} + \Delta_i$ is nef for $i \gg 0$ and the program stops. If $\lambda > 0$, choose $0 < \mu < \lambda$. Then every $(K_{X_{i_0}} + \Delta_{i_0})$-flip is a $(K_{X_{i_0}} + \Delta_{i_0} + \mu A_{i_0})$-flip, and every sequence of $(K_{X_{i_0}} + \Delta_{i_0} + \mu A_{i_0})$-flips with scaling of A_{i_0} is finite by Theorem 6.5. □

7. Abundance

Theorem 7.1. *Conjecture 1.3 implies log abundance. In other words, if (X, Δ) is a projective klt pair and $K_X + \Delta$ is nef, then $K_X + \Delta$ is semiample.*

Proof. The proof is almost verbatim the proof of Corollary 3.9. The only difference is at the beginning of the proof when, instead of invoking Theorem 3.2, we invoke Conjecture 1.3. □

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